Auslander-Reiten Quivers as a Cluster Algebra of Type A_n

10/23/24

I. Intro to Cluster Algebras

Def: given a quiver Q, $v \in Q$, can mutate at v to obtain a new quiver

(1) $\forall\, i \to v \to j$, add an arrow $i \to j$

(2) remove all 2-cycles

(3) flip all arrows adjacent to v

Def: we associate a variable x_v to each vertex $v \in Q$, and mutation at x_v replaces x_v with x_v' s.t. $x_v x_v' = \prod_{i \to v} x_i + \prod_{v \to j} x_j$

- we call the $\{x_v : v \in Q\}$ cluster variables
- the set of cluster variables in a given quiver form a cluster
- the cluster algebra A_Q over $\mathbb{R}$ (usually $\mathbb{C}$) is generated by all cluster variables across all mutations

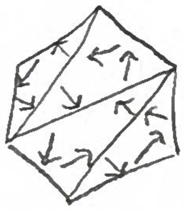
(Fomin+Zelevinsky, 2002)

Thm: a cluster algebra is of finite type $\Leftrightarrow$ is a quiver mutation equiv to a Dynkin diagram

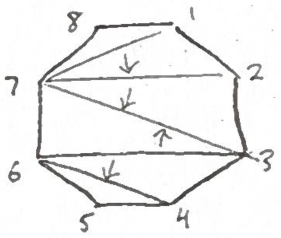
Type A_n: $\underset{1}{\bullet} - \underset{2}{\bullet} - \cdots - \underset{n}{\bullet}$

Thm: $\{$clusters of type $A_n\} \longleftrightarrow \{$triangulations of an $(n+3)$-gon$\}$

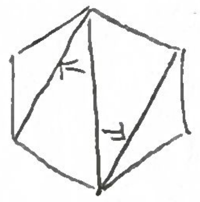
mutation $\longleftrightarrow$ flips of diagonals



$\longleftrightarrow$ $\bullet \rightarrow \bullet \leftarrow \bullet$

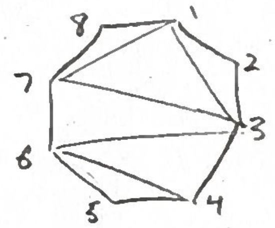


$\longleftrightarrow$



$\longleftrightarrow$ $\bullet \leftarrow \bullet \rightarrow \bullet$

mutate at 27



$\longleftrightarrow$

II. The Cluster Category

"the Knitting Algorithm": Given an orientation of a simply laced Dynkin diagram, Q

$\mathbb{Z}Q$ is $\mathbb{Z}$ copies of Q drawn so arrows go right

(1) $\forall\, \alpha : i \to j$ in Q, add arrows $(n,j) \to (n+1,i)$

(2) Define $\tau : \mathbb{Z}Q \to \mathbb{Z}$

$(p,i) \mapsto (p-1,i)$ $\forall p \in \mathbb{Z}$

$(p,\alpha) \mapsto (p-1,\alpha)$

the Path Category kQ:

- objects are vertices of Q
- morphisms are $\mathrm{Hom}(i,j) = \mathrm{Span}\{$paths $i \to j\}$
- composition = composition of paths

Mesh Relations: given a mesh

$$\begin{tikzcd} & u_1 \arrow[dr, "\alpha_1^*"] & \\ \tau(v) \arrow[ur, "\alpha_1"] \arrow[r] \arrow[dr] & u_2 \arrow[r] & v \\ & u_s \arrow[ur] & \end{tikzcd}$$

in $k\mathbb{Z}Q$, let $\sum_{i=1}^{s} \alpha_i \alpha_i^* = 0$

Note: For simply laced Q, we have $\mathbb{Z}Q$ is periodic

Rule for $\underline{\dim}$: $\underline{\dim} M_v + \underline{\dim} M_{\tau v} = \sum_{i=1}^{s} \underline{\dim} M_{u_i}$

(each vertex is secretly a right kQ-module)

Ex: $1 \to 2 \to 3 \to 4$ A_4

$P_i = \{$all paths ending at $i\}$, projective right module

$\underline{\dim} P_i = (v_1, v_2, v_3, v_4)$ where $v_j = \begin{cases} 1 & \text{if } \exists \text{ path } j \to i \\ 0 & \text{otherwise} \end{cases}$

$\underline{\dim} P_1 = (1,0,0,0) = 1$

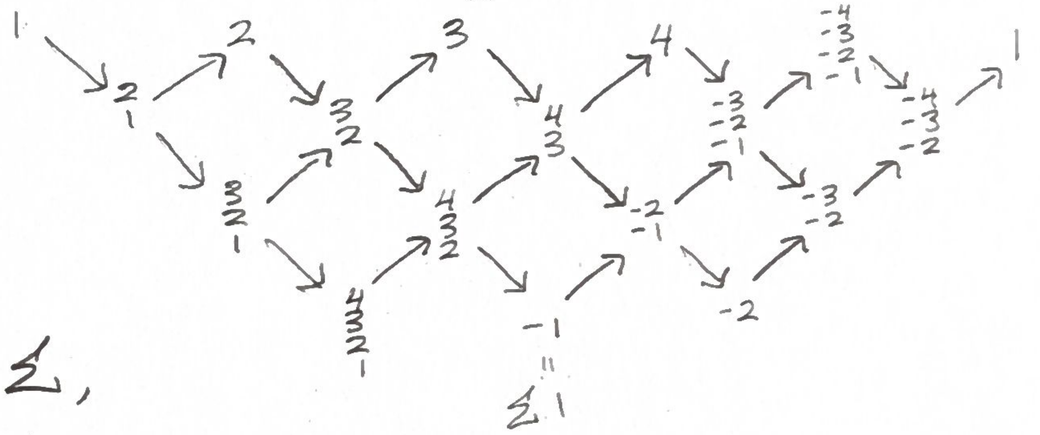
$\underline{\dim} P_2 = (1,1,0,0) = \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}$

$\underline{\dim} P_3 = (1,1,1,0) = \begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix}$

$\underline{\dim} P_4 = (1,1,1,1) = \begin{smallmatrix} 4 \\ 3 \\ 2 \\ 1 \end{smallmatrix}$

"A-R quiver"

all + or all − ⟺ roots in Φ



Fact: Mesh ($\mathbb{Z}Q$), with shift Σ, forms a triangulated category

Def: the cluster category is $\cong \dfrac{\text{Mesh}(\mathbb{Z}Q)}{\tau^{-1}\Sigma}$

cluster variables ⟷ vertices of A-R quiver

clusters ⟷ cluster-tilting sets $\{T_1, \dots, T_n\}$ s.t. $\forall i,j$, $\text{Hom}(T_i, \Sigma T_j) = 0$

mutation ⟷ mutation via cones of triangles

Def: (endomorphism quiver)

Given a cluster-tilting set $\{T_1, \dots, T_n\}$,

- vertices are each T_i
- # arrows from $i \to j$ = $\dim(\text{Hom}(T_i, T_j))$

Mutation at T_k: Replace T_k with $T_k^* \cong {}^*T_k$

(1) $T_k \xrightarrow{u} \bigoplus_{k \to i} T_i \to T_k^* \to \Sigma T_k$

(2) ${}^*T_k \to \bigoplus_{j \to k} T_j \to T_k \to \Sigma\, {}^*T_k$

$$\begin{tikzcd} & E_1 \arrow[dr] & \\ X \arrow[ur] \arrow[dr] & & Y \\ & E_2 \arrow[ur] & \end{tikzcd}$$

gives Δ $X \to E_1 \oplus E_2 \to Y \to \Sigma X$

Auslander-Reiten Quivers as a Cluster Algebra of Type A_n

10/23/24

Ex: $1 \oplus \begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\\1\end{smallmatrix} \oplus \begin{smallmatrix}4\\3\\2\\1\end{smallmatrix} \overset{\mu_{\begin{smallmatrix}2\\1\end{smallmatrix}}}{\longmapsto} 1 \oplus ? \oplus \begin{smallmatrix}3\\2\\1\end{smallmatrix} \oplus \begin{smallmatrix}4\\3\\2\\1\end{smallmatrix}$

Endoquiver: $1 \to \begin{smallmatrix}2\\1\end{smallmatrix} \to \begin{smallmatrix}3\\2\\1\end{smallmatrix} \to \begin{smallmatrix}4\\3\\2\\1\end{smallmatrix}$

$\begin{smallmatrix}2\\1\end{smallmatrix} \to \begin{smallmatrix}3\\2\\1\end{smallmatrix} \to T_k^* \to \Sigma \begin{smallmatrix}2\\1\end{smallmatrix} \Rightarrow T_k^* = 3$

$1 \oplus 3 \oplus \begin{smallmatrix}3\\2\\1\end{smallmatrix} \oplus \begin{smallmatrix}4\\3\\2\\1\end{smallmatrix} \overset{\mu_{\begin{smallmatrix}3\\2\\1\end{smallmatrix}}}{\longmapsto} 1 \oplus 3 \oplus ? \oplus \begin{smallmatrix}4\\3\\2\\1\end{smallmatrix}$

Endoquiver:

$$\begin{tikzcd} 1 \arrow[rr, bend right, "\beta"'] & 3 \arrow[l, "\alpha"'] & \begin{smallmatrix}3\\2\\1\end{smallmatrix} \arrow[l, "\gamma"'] \arrow[r] & \begin{smallmatrix}4\\3\\2\\1\end{smallmatrix} \end{tikzcd}$$

where $\beta\alpha = \gamma\beta = \alpha\gamma = 0$

$\begin{smallmatrix}3\\2\\1\end{smallmatrix} \to 3 \oplus \begin{smallmatrix}4\\3\\2\\1\end{smallmatrix} \to T_k^* \to \Sigma \begin{smallmatrix}3\\2\\1\end{smallmatrix} \Rightarrow T_k^* = \begin{smallmatrix}4\\3\end{smallmatrix}$

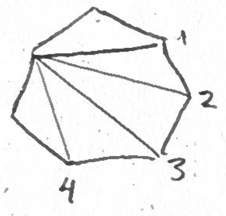
Endoquiver:

$$\begin{tikzcd} 1 \arrow[rrr, bend right] & 3 \arrow[l, bend left] \arrow[r] & \begin{smallmatrix}4\\3\end{smallmatrix} & \begin{smallmatrix}4\\3\\2\\1\end{smallmatrix} \arrow[l] \end{tikzcd}$$

III. Bijection with Triangulations

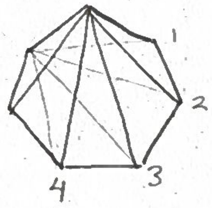
Working Example: $1 \to 2 \to 3 \to 4$

Reference Triangulation:
(any triangulation with this quiver)

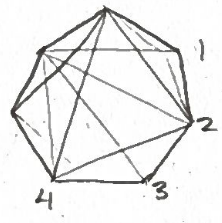


Note: $\Sigma = \tau =$ rotation ccw
So, original triangulation corresponds to $\Sigma 1 \oplus \Sigma \begin{smallmatrix}2\\1\end{smallmatrix} \oplus \Sigma \begin{smallmatrix}3\\2\\1\end{smallmatrix} \oplus \Sigma$
works for all orientations of A_n

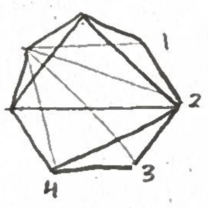
$1 \oplus \begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\\1\end{smallmatrix} \oplus \begin{smallmatrix}4\\3\\2\\1\end{smallmatrix} \longleftrightarrow$



$1 \oplus 3 \oplus \begin{smallmatrix}3\\2\\1\end{smallmatrix} \oplus \begin{smallmatrix}4\\3\\2\\1\end{smallmatrix} \longleftrightarrow$



$1 \oplus 3 \oplus \begin{smallmatrix}4\\3\end{smallmatrix} \oplus \begin{smallmatrix}4\\3\\2\\1\end{smallmatrix} \longleftrightarrow$



IV. Bijection with Almost Positive Roots

$1 \longleftrightarrow \alpha_1 = e_1 - e_2$

$\begin{smallmatrix}2\\1\end{smallmatrix} \longleftrightarrow \alpha_1 + \alpha_2 = e_1 - e_3$

$\begin{smallmatrix}3\\2\\1\end{smallmatrix} \longleftrightarrow \alpha_1 + \alpha_2 + \alpha_3 = e_1 - e_4$

$\begin{smallmatrix}4\\3\\2\\1\end{smallmatrix} \longleftrightarrow \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = e_1 - e_5$

each module $\longleftrightarrow$ linear expansion into simple roots

$\tau^{-1} \longleftrightarrow$ application of corresponding Coxeter elt to that root

Φ_+ then Φ_- gives periodic A-R quiver