11/20/25

Catalan Numbers

I. History

(1) 1730: 明安图 astronomer proved $\sin(2\alpha) = 2\sin\alpha - \sum_{n=1}^{\infty} \frac{C_{n-1}}{4^{n-1}} \sin^{2n+1}\alpha$

(2) 1751: Euler (w/ Goldbach + Segner) $C_n = \#$ triangulations of $(n+2)$-gon

(3) 1838: Catalan ← French $\quad C_n = \#$ bracketings of $n+1$ letters

<u>Name</u>: "Segner numbers", "Euler-Segner numbers"

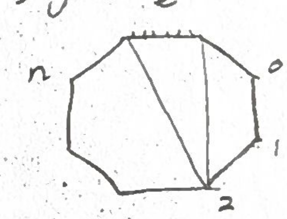
John Riordan ← American: Math Reviews (1948) (1964)

- 1968 in "Combinatorial Identities"

II. Combinatorial Interpretations : Triangulations of $(n+2)$-gon

(1) Recurrence: $C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}, \quad C_0 = 1$

↑ which pt completes Δ including e?



<u>Solving for C_n</u>: Generating Functions $\quad y = \sum_{n\geq 0} C_n x^n$

from recurrence, $\sum_{n\geq 0} C_{n+1} x^n = \sum_{n\geq 0} \left(\sum_{k=0}^{n} C_k C_{n-k} \right) x^n$

(1) $\sum_{n\geq 0} C_{n+1} x^{n+1} = y - 1$, so $LHS = \frac{y-1}{x}$

(2) $\sum_{k=0}^{n} C_k C_{n-k}$ is coef of x^n in $\left(\sum_{n\geq 0} C_n x^n\right)^2$, so $RHS = y^2$

$\Rightarrow$ Solve for y: $\frac{y-1}{x} = y^2 \Rightarrow xy^2 - y + 1 = 0 \Rightarrow y = \frac{1 \pm \sqrt{1-4x}}{2x}$

$+$ doesn't make sense, so we have

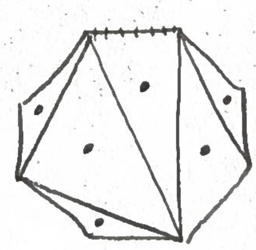
$$y = \frac{1-\sqrt{1-4x}}{2x} = \frac{1}{2x} - \frac{1}{2x}\sum_{n\geq 0}\binom{\frac{1}{2}}{n}(-4x)^n$$

↑ binomial formula

$$= \sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n} x^n$$

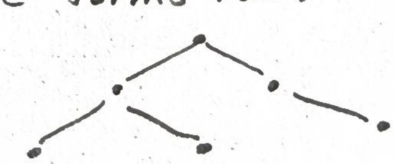
(2) Binary Trees on n vertices

$\longleftrightarrow$ Triangulations



e forms root node

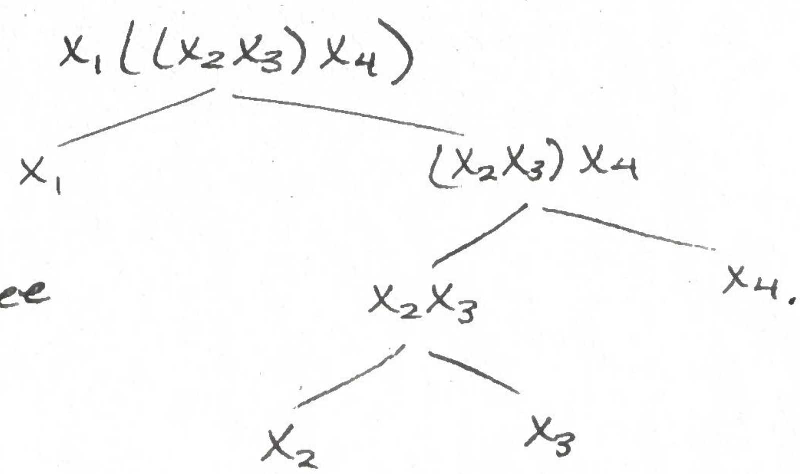
nodes are Δ



(3) Binary bracketings of $n+1$ letters

$\longleftrightarrow$ binary trees

- must apply operation n times
- $2n+1$ nodes, where $n+1$ are x_i
- remove these leaves to get binary tree

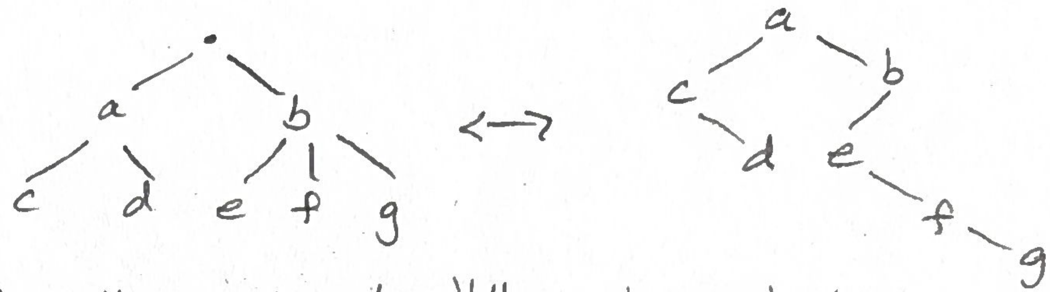


(4) Plane trees on $n+1$ nodes: children are ordered left to right.

$\longleftrightarrow$ binary trees

(i) ignore root

(ii) for every parent / children, keep only eldest relationship, then connect siblings by seniority



In the end, a left child is a true child, and a right child is a younger sibling

"restructuring based on seniority"

(5) Ballot Sequences: A, B get n votes each

a counting of votes s.t. A never trails B

$\longleftrightarrow$ plane trees   take DFS ordering, breaking ties by age.

(i) step up $\longleftrightarrow +1$

(ii) step down $\longleftrightarrow -1$

$1, 1, -1, -1, 1, -1$

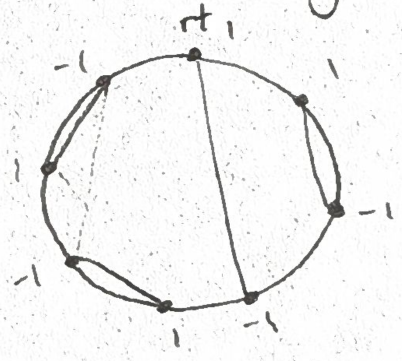
"trees grow up" - Igor Pak

(6) Dyck Paths: length $2n$ NE lattice paths $(0,0)$ to (n,n) that never fall below $y=x$.

$\longleftrightarrow$ ballot sequences

(7) Noncrossing Chords: n chords connecting $2n$ distinct pts on circle

$\longleftrightarrow$ ballot sequences:

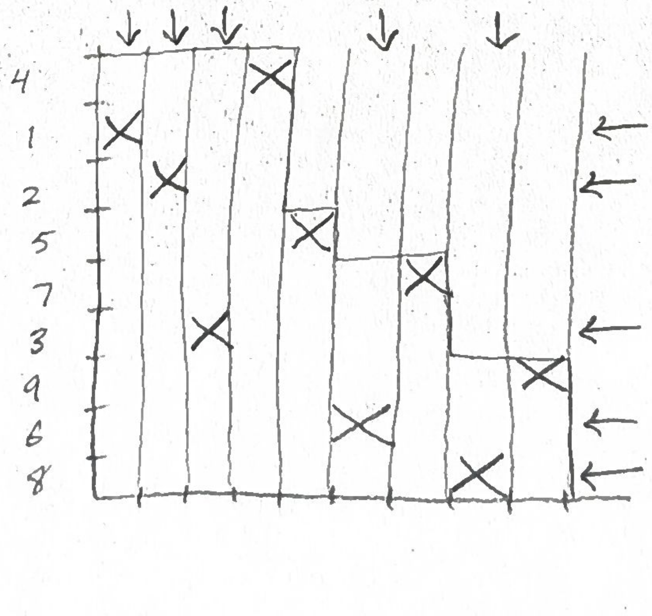


Fix a root.
Traveling clockwise,
each new chord is $+1$
each old chord is -1.

(8) 321-avoiding Permutations:

there does not exist $i<j<k$ s.t. $a_i > a_j > a_k$

$\longleftrightarrow$ Dyck paths $\sigma = 412573968 \in S_9$



To reverse the bijection:

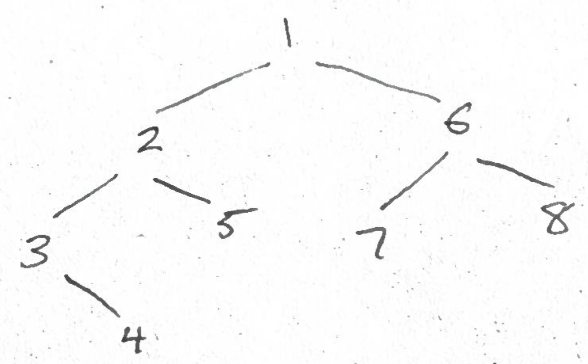
(1) place an x at every peak in the Dyck path

(2) identify all rows + columns that have not yet been used

(3) place X at each intersection in increasing order

(9) 312-avoiding permutations of $[n]$: $\neg\exists$ $i<j<k$ s.t. $a_j < a_k < a_i$.
"pattern avoidance" Ex: 34251768

Bijection with binary trees:
(1) Remove lowest letter, split word into 2 children
(2) Repeat
(3) get a labelled binary tree s.t. DFS gives $1\,2 \cdots n$



Note: will get a binary tree regardless. (3) holds because whenever we split, everything on the left child must be less than everything on right child. Is easy to reconstruct permutation from tree.

Rmk: 321-avoiding permutations of $[n]$ also in bijection w/ Dyck paths

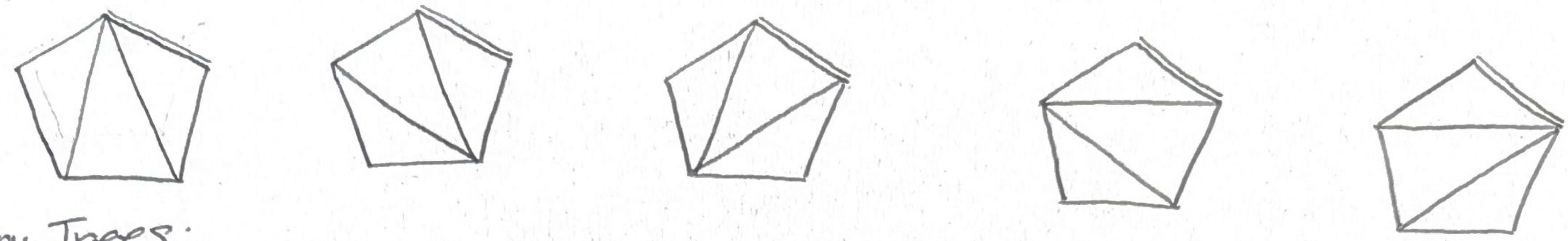
Pf sketch of direct combinatorial pf of $\frac{1}{n+1}\binom{2n}{n} = C_n$:
via Ballot sequences:
(1) Given a sequence with $n+1$ 1's and n -1's, exactly one cyclic shift has property that $a_i = 1$ and $a_{i+1}, \ldots, a_{i-1}$ is a ballot seq.
(2) there are $n+1$ cyclic shifts beginning w/ 1. Only 1 gives ballot seq. when removing 1st term
(3) there are $\binom{2n}{n}$ sequences of $n+1$ 1's, n -1's s.t. $a_0 = 1$.
$\Rightarrow$ # ballot seq $= \frac{1}{n+1}\binom{2n}{n}$.

Induction on (1):
Base: $(1,1,-1)$ $(1,-1,1)$ $(-1,1,1)$ ✓
IS: existance: $\exists$ a shift s.t. $1,-1$ occurs. Ignore, then $!\exists$ shift satisfying IH.
Put $(1,-1)$ back in and this is a ballot seq. ✓
uniqueness: if $\exists$ another pair $(1,-1)$, this cannot take 1st 2 places of the unique shift, and therefore doesn't affect any properties satisfied.
So by IH switching choice of $(1,-1)$ cannot affect the (being a ballot seq) unique cyclic shift ✓.

Note: this talk was adapted from Richard Stanley's talk and Igor Pak's history of the Catalan numbers.

Table of Combinatorial objects for $C_3 = 5$:

Triangulations:

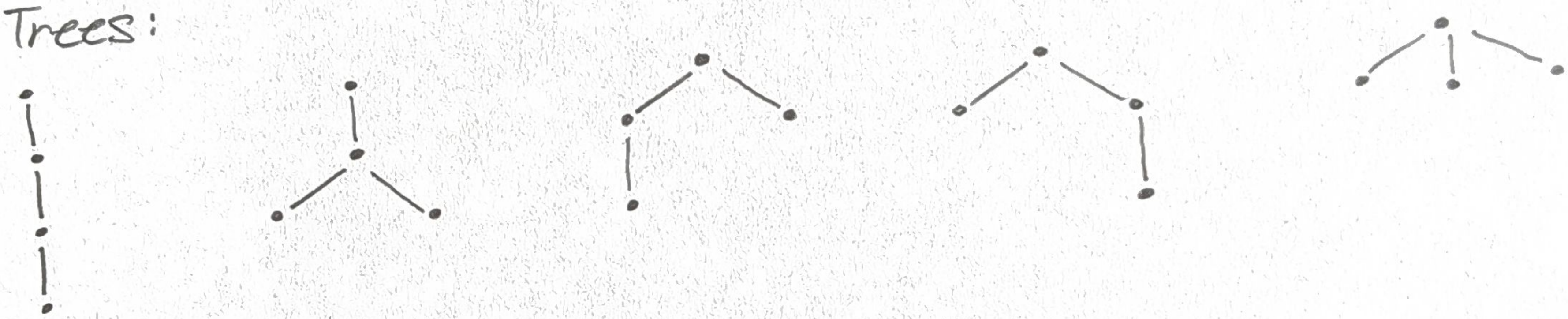


Binary Trees:

Bracketings:

$((x_1x_2)x_3)x_4$ $(x_1(x_2x_3))x_4$ $(x_1x_2)(x_3x_4)$ $x_1((x_2x_3)x_4)$ $x_1(x_2(x_3x_4))$

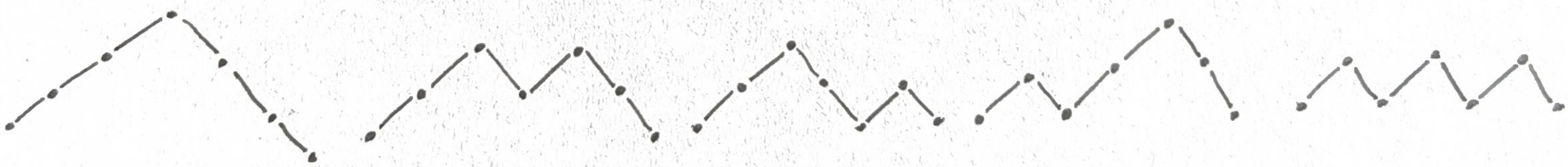
Plane Trees:



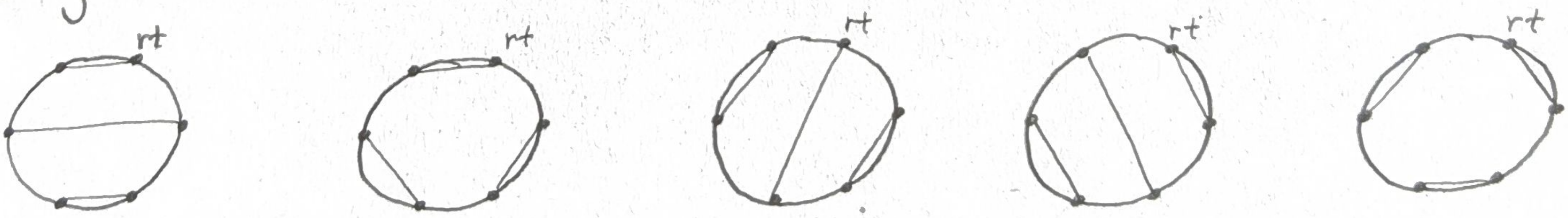
Ballot Sequences:

$(1,1,1,-1,-1,-1)$ $(1,1,-1,1,-1,-1)$ $(1,1,-1,-1,1,-1)$ $(1,-1,1,1,-1,-1)$ $(1,-1,1,-1,1,-1)$

Dyck Paths:



Noncrossing Chords:



312-avoiding Permutations:

321 231 213 132 123

321-avoiding Permutations:

312 231 213 132 123