Math 131A: Analysis
Discussion 10: Integrability

1. Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable on $[a, b]$.

   (a) Show that if one value of $f(x)$ is changed at some point $x \in [a, b]$, this does not affect the integral and integrability of f.
   Solution:
   Let $\epsilon > 0$. Let $g(x)$ be the new function, defined as $f(x)$ everywhere except for $g(x_0) \neq f(x_0)$.
   Since f is integrable, there exists a partition $P = \{a = t_0, t_1, \ldots, t_n = b\}$ such that $U(f, P) - L(f, P) < \frac{\epsilon}{2}$.
   Let $M = |g(x_0) - f(x_0)|$ be the change in the value of $f(x_0)$. Let P' be the refinement of P defined by

$$P' = P \cup \{x_0 - \frac{\epsilon'}{2}, x_0 + \frac{\epsilon'}{2}\} = \{s_0 < s_1 < \cdots < s_{n+2}\}$$

   where $\epsilon' = \frac{\epsilon}{2M}$, and $s_\alpha = x_0 - \frac{\epsilon'}{2}$, $s_\beta = x_0 + \frac{\epsilon'}{2}$. Then we have

$$\begin{aligned}
U(g, P') - L(g, P') &= \sum_{i=1}^{n+2} osc(g, [s_{i-1}, s_i]) \cdot (s_i - s_{i-1}) \\
&= \sum_{1 \le i \le \alpha \text{ or } \beta+1 \le i \le n+2} osc(f, [s_{i-1}, s_i]) \cdot (s_i - s_{i-1}) + \sum_{i=\alpha+1}^{\beta} osc(g, [s_{i-1}, s_i]) \cdot (s_i - s_{i-1}) \\
&\le \sum_{1 \le i \le \alpha \text{ or } \beta+1 \le i \le n+2} osc(f, [s_{i-1}, s_i]) \cdot (s_i - s_{i-1}) + \sum_{i=\alpha+1}^{\beta} \big(osc(f, [s_{i-1}, s_i]) + M\big) \cdot (s_i - s_{i-1}) \\
&= \big(U(f, P') - U(f, P')\big) + M(s_\beta - s_\alpha) \\
&\le \big(U(f, P) - U(f, P)\big) + M\epsilon' \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

   (b) Show this remains true if we change a finite number of values
   Solution: Call the new function $g(x)$, which is the same as $f(x)$ everywhere except for $x_1, \ldots, x_k$. Define a new function $h(x) = g(x) - f(x)$. It is sufficient to show that $h(x)$ is Riemann integrable, as $g(x) = h(x) + f(x)$ the sum of two integrable functions is integrable. Notice that $h(x) = 0$ almost everywhere.
   Let $\epsilon > 0$. Let $M = \max_i |g(x_i) - f(x_i)|$.
   Construct a partition

$$P = \left\{a, b, x_1 \pm \frac{\epsilon'}{2}, \ldots, x_k \pm \frac{\epsilon'}{2}\right\} = \{t_0 < t_1 < \cdots < t_n\}$$

where $\epsilon' = \frac{\epsilon}{2kM}$. Then,

$$\begin{aligned}
U(h,P) - L(h,P) &= \sum_{i=1}^{n} osc(h, [t_{i-1}, t_i]) \cdot (t_i - t_{i-1}) \\
&= \sum_{i=1}^{k} osc\left(h, \left[x_i - \frac{\epsilon'}{2}, x_i + \frac{\epsilon'}{2}\right]\right) \cdot \epsilon' \\
&\leq \sum_{i=1}^{k} M\epsilon' \\
&= kM\frac{\epsilon}{2kM} = \frac{\epsilon}{2} < \epsilon
\end{aligned}$$

So, $h(x)$ is Riemann integrable, and thus so is $g(x)$.

(c) Find an example to show that altering f at an infinite number of points may cause the resulting function to no longer be Riemann integrable.
Solution:
Change the function to be

$$g(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Notice that for every partition P,

$$U(f,P) = 1 \cdot (b-a) \neq 0 \cdot (b-a) = L(f,P).$$

So, $g(x)$ is not integrable over any $[a,b]$.

2. Show that the function $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & x = \frac{1}{2^n}, n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

is Riemann integrable.
Solution:
Let $\epsilon > 0$. Notice that there are only finitely many discontinuities outside of $[0, \frac{\epsilon}{4}]$. Call them $x_1, \dots, x_k$. Then, define the partition

$$P = \left\{0, \frac{\epsilon}{4}, 1, x_1 \pm \frac{\epsilon'}{2}, \dots, x_k \pm \frac{\epsilon'}{2}\right\} = \{t_0 < t_1 < \cdots < t_n\}$$

where $\epsilon' = \frac{\epsilon}{2k}$. Then,

$$\begin{aligned}
U(f,P) - L(f,P) &= \sum_{i=1}^{n} osc(f, [t_{i-1}, t_i]) \cdot (t_i - t_{i-1}) \\
&= osc(f, [0, \frac{\epsilon}{4}]) \cdot \frac{\epsilon}{4} + \sum_{i=1}^{k} osc\left(f, \left[x_i - \frac{\epsilon'}{2}, x_i + \frac{\epsilon'}{2}\right]\right) \cdot \epsilon' \\
&\leq \frac{\epsilon}{4} + \sum_{i=1}^{k} \epsilon' \\
&= \frac{\epsilon}{4} + k\frac{\epsilon}{2k} \\
&= \frac{3\epsilon}{4} < \epsilon
\end{aligned}$$

So, $f(x)$ is Riemann integrable on $[0,1]$.