Math 131A: Analysis
Discussion 3: Supremum and Infimum, Sequences

1 Supremum and Infimum

Note: Clarification on homeworks: statements vs sets

Definition: Let $S \subset \mathbb{R}$ be a nonempty subset.
If S contains an element $m \in S$ such that $\forall s \in S$, $s \leq m$, then m is the *maximum* of S. Similarly, if S contains an element $\ell \in S$ such that $\forall s \in S$, $s \geq \ell$, then ℓ is the *minimum* of S.

Definition: Let $S \subset \mathbb{R}$ be a nonempty subset.
If S is bounded above and S has a least upper bound (there is no smaller upper bound), we call it the *supremum* of S, $\sup S$. Similarly, if S is bounded below and S has a greatest lower bound (there is no larger lower bound), we call it the *infimum* of S, $\inf S$.

1. If they exist, find the supremum, infimum, maximum, and minimum of the following sets, viewed as subsets of $\mathbb{Q}$.
   (i) $\{\frac{1}{n} : n \in \mathbb{N}\}$
   (ii) $(0, 1)$
   (iii) $\{\pi, e\}$
   (iv) $\{r \in \mathbb{Q} : r < 2\}$
   (v) $\{r \in \mathbb{Q} : r^2 < 2\}$
2. (4.5) Let S be a nonempty bounded subset of $\mathbb{R}$. Prove if $\sup S \in S$, then $\sup S = \max S$.
3. (4.7) Let S and T be nonempty bounded subsets of $\mathbb{R}$.
   (a) Prove if $S \subseteq T$, then $\inf T \leq \inf S \leq \sup S \leq \sup T$.
   (b) Prove $\sup(S \cup T) = \max\{\sup S, \sup T\}$.
4. (4.12) Let $\mathbb{I}$ be the set of irrational numbers. Prove if $a < b$, then there exists an $x \in \mathbb{I}$ such that $a < x < b$.
5. (4.14) Let A, B nonempty bounded subsets of $\mathbb{R}$, and let $A + B = \{a + b : a \in A, b \in B\}$.
   (a) Prove $\sup(A + B) = \sup A + \sup B$.
   (b) Prove $\inf(A + B) = \inf A + \inf B$.

2 Sequences

Definition: A sequence (s_n) of real numbers *converges* to the real number s if
$\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n > N$, $|s_n - s| < \epsilon$. This s is called the *limit* of (s_n). If the sequence does not converge to a real number, then it *diverges*.

1. Which of these sequences converge?
   (i) $a_n = \frac{n^2+3}{4n^2-3}$
   (ii) $a_n = n!$
   (iii) $a_n = \frac{(-1)^n}{n}$
   (iv) $a_n = 73 + (-1)^n$
   (v) $a_n = \sin(n\pi)$
2. Give an example of a sequence of irrational numbers having a limit that is a rational number
3. Given an example of a sequence of rational numbers having a limit that is an irrational number