

Math 131A: Analysis

Discussion 4: Convergence, Limits, and Cauchy sequences

1 Sequences and Limits

Definition: A sequence (s_n) of real numbers *converges* to the real number s if $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $\forall n > N, |s_n - s| < \epsilon$. This s is called the *limit* of (s_n) . If the sequence does not converge to a real number, then it *diverges*.

1. Which of these sequences converge?

- (i) $a_n = \frac{n^2+3}{4n^2-3}$
- (ii) $a_n = n!$
- (iii) $a_n = \frac{(-1)^n}{n}$
- (iv) $a_n = 73 + (-1)^n$
- (v) $a_n = \sin(n\pi)$

2. Which of these sequences are increasing? Decreasing? Bounded?

- (i) $\frac{1}{n}$
- (ii) $\frac{(-1)^n}{n^2}$
- (iii) $\sin(\frac{n\pi}{7})$
- (iv) $\frac{n}{3^n}$

3. (a) Let $(a_n), (b_n), (c_n)$ be sequences such that $b_n \leq a_n \leq c_n$ for all $n \in \mathbb{N}$, and

$$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} c_n = a$$

Show that $\lim_{n \rightarrow \infty} a_n = a$.

(b) Let $S \subset \mathbb{R}$ nonempty subset bounded above. Show there exists a sequence (a_n) such that $\lim_{n \rightarrow \infty} a_n = \sup S$.

4. Suppose $(a_n), (b_n)$ are sequences such that $\lim_{n \rightarrow \infty} a_n = 0$ and (b_n) is a bounded sequence. Show that

$$\lim_{n \rightarrow \infty} a_n b_n = 0$$

5. (Density)

- (a) Show that the density of $\mathbb{Q} \subset \mathbb{R}$ implies $\forall x \in \mathbb{R}$, there exists a sequence $(p_n) \subset \mathbb{Q}$ such that $\lim_{n \rightarrow \infty} p_n = x$
- (b) Show that the irrationals are dense in $\mathbb{R}$: for any $x \in \mathbb{R}$, show there exists a sequence (q_n) of irrationals such that $\lim_{n \rightarrow \infty} q_n = x$.

2 Monotone and Cauchy Sequences

Theorem: All bounded monotone sequences converge

Definition: A sequence $(s_n) \subset \mathbb{R}$ is called a *Cauchy sequence* if $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that

$$m, n > N \implies |s_n - s_m| < \epsilon$$

Theorem: A sequence is a convergent sequence iff it is a Cauchy sequence

1. Let $s_1 = 1$ and $s_{n+1} = \frac{1}{3}(s_n + 1)$ for $n \geq 1$.

- (a) Find s_2, s_3, s_4
- (b) Use induction to show $s_n > \frac{1}{2}$ for all n .
- (c) Show (s_n) is decreasing
- (d) Show $\lim s_n$ exists, and find it

2. Let $t_1 = 1$ and $t_{n+1} = \left(1 - \frac{1}{(n+1)^2}\right) \cdot t_n$ for $n \geq 1$

- (a) Show $\lim t_n$ exists
- (b) Use induction to show $t_n = \frac{n+1}{2n}$
- (c) What is $\lim t_n$?

3. (10.6)

- (a) Let (s_n) be a subsequence such that

$$|s_{n+1} - s_n| < 2^{-n}, \forall n \in \mathbb{N}$$

Prove (s_n) is a Cauchy sequence and hence a convergent sequence.

- (b) Is the result in (a) true if we only assume $|s_{n+1} - s_n| < \frac{1}{n}$ for all $n \in \mathbb{N}$?

4. (10.4) Discuss why the above theorems fail if we restricted to the set $\mathbb{Q}$ of rational numbers