Math 131A: Analysis
Discussion 6: Cauchy sequences and Series

1 Cauchy Sequences and Subsequences

Definition: A sequence $(s_n) \subset \mathbb{R}$ is called a *Cauchy sequence* if $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$m, n > N \implies |s_n - s_m| < \epsilon$$

Theorem: A sequence is a convergent sequence iff it is a Cauchy sequence
Definition: Let (s_n) be a sequence in $\mathbb{R}$. A *subsequential limit* is any real number or symbol $\pm\infty$ that is the limit of some subsequence of (s_n).

1. (10.6)
   (a) Let (s_n) be a subsequence such that
   $$|s_{n+1} - s_n| < 2^{-n}, \forall n \in \mathbb{N}$$
   Prove (s_n) is a Cauchy sequence and hence a convergent sequence.
   (b) Is the result in (a) true if we only assume $|s_{n_1} - s_n| < \frac{1}{n}$ for all $n \in \mathbb{N}$?
2. (10.4) Discuss why the above theorems fail if we restricted to the set $\mathbb{Q}$ of rational numbers
3. For each sequence, give its set of subsequential limits. What is the lim sup and lim inf?
   (a) $a_n = (-1)^n$
   (b) $b_n = \frac{1}{n}$
   (c) $c_n = n^2$
   (d) $d_n = \frac{6n+4}{7n-3}$
4. Does there exist a...
   (a) Cauchy sequence that is not monotone
   (b) Monotone sequence that is not Cauchy
   (c) Monotone sequence that diverges but has a convergent subsequence
   (d) An unbounded sequence that contains a convergent subsequence
   (e) A sequence that contains subsequences converging to every point in the infinite set $(1, \frac{1}{2}, \frac{1}{3}, \dots)$

2 Convergence of Infinite Series

Definition: Let $(a_n)_{n\geq m}$ be a sequence. The sequence of partial sums (s_n) is defined as $s_n = \sum_{k=m}^{n}$. The infinite series $\sum_{k=m}^{\infty} a_k$ *converges* if its sequence of partial sums converges, and *diverges* otherwise.

Comparison Test: Let $\sum a_n$ be a series where $a_n \geq 0$ for all n.

(i) If $\sum a_n$ converges and $|b_n| \leq a_n$ for all n, then $\sum b_n$ converges.

(ii) If $\sum a_n = \infty$ and $b_n \geq a_n$ fpr a;; n, then $\sum b_n = \infty$.

Ratio Test: Let $\sum a_n$ be a series of nonzero terms. The series

(i) converges absolutely if $\limsup |a_{n+1}/a_n| < 1$

(ii) diverges if $\liminf |a_{n+1}/a_n| > 1$

(iii) otherwise, $\liminf |a_{n+1}/a_n| \leq 1 \leq \limsup |a_{n+1}/a_n|$ and the test gives no information

Root Test: Let $\sum a_n$ be a series of nonzero terms, and let $\alpha = \limsup |a_n|^{1/n}$. The series

(i) converges absolutely if $\alpha < 1$

(ii) diverges if $\alpha > 1$

(iii) otherwise, $\alpha = 2$ and the test gives no information

1. Which of the following series converge?

   (a)
   $$\sum_{n=2}^{\infty} \left(-\frac{1}{3}\right)^n$$

   (b)
   $$\sum \frac{n}{n^2+3}$$

   (c)
   $$\frac{1}{n^2+1}$$

   (d)
   $$\sum \frac{n!}{n^n}$$

   (e)
   $$\sum \frac{n^4}{2^n}$$

   (f)
   $$\sum \frac{2^n}{n!}$$