Math 131A: Analysis
Discussion 7: Series Convergence and Limits of Functions

1 Convergence of Infinite Series

Ratio Test: Let $\sum a_n$ be a series of nonzero terms. The series

(i) converges absolutely if $\limsup |a_{n+1}/a_n| < 1$

(ii) diverges if $\liminf |a_{n+1}/a_n| > 1$

(iii) otherwise, $\liminf |a_{n+1}/a_n| \leq 1 \leq \limsup |a_{n+1}/a_n|$ and the test gives no information

Alternating Series Test: Let (a_n) be a decreasing sequence that converges to 0. Then, the series

$$\sum_{n=0}^{\infty} (-1)^n a_n$$

converges.

p-Series Test: $\sum_{n=0}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

1. Which of the following series converge?

   (a)
   $$\sum_{n=2}^{\infty} \left(-\frac{1}{3}\right)^n$$

   (b)
   $$\frac{1}{n^2+1}$$

   (c)
   $$\sum \frac{n!}{n^n}$$

   (d)
   $$\sum \frac{n^4}{2^n}$$

   (e)
   $$\sum \frac{2^n}{n!}$$

   (f)
   $$\sum \frac{(-1)^n n!}{2^n}$$

2. (a) Give an example of a divergent series $\sum a_n$ for which $\sum a_n^2$ converges

   (b) Give an example of a convergent series $\sum a_n$ for which $\sum a_n^2$ diverges

2 Limits of Functions

Definition: Let $A \subset \mathbb{R}$. $c \in \mathbb{R}$ is a *limit point* of A if there exists a sequence $(a_n) \subset A - \{c\}$ such that

$$\lim_{n\to\infty} a_n = c.$$

A set is *closed* if it contains all its limit points

Definition: Let $A \subset \mathbb{R}$, and $c \in A$ a limit point of A. Then, $\lim_{x\to c} = L$ if for every sequence $\lim a_n \to c$ converging to c, we have $\lim_{n\to\infty} f(a_n) = L$.

Definition: The *right hand limit* of $f(x)$ at c is $\lim_{x\to c^+} f(x) = R$, and exists if for every sequence $a_n > c$ converging to c, $\lim_{n\to\infty} f(a_n) = R$.
The *left hand limit* of $f(x)$ at c is $\lim_{x\to c^-} f(x) = L$, and exists if for every sequence $a_n < c$ converging to c, $\lim_{n\to\infty} f(a_n) = L$.

1. Is $[0,1] \cap \mathbb{Q}$ closed in the rational numbers? What about the real numbers?
2. Is the set of irrational numbers closed in $\mathbb{R}$?
3. For the following functions, what are the right and left limits at 0? What are the limits at $\pm\infty$?
   (i) $f(x) = \frac{x^3}{|x|}$
   (ii) $f(x) = \frac{\sin x}{x}$
   (iii) $f(x) = \sin \frac{1}{x}$
4. Let $A \subset \mathbb{R}$. Let $f : A \to \mathbb{R}$ and $c \in \mathbb{R}$ a limit point of A. Prove the following:

$$\lim_{x\to c} f(x) = L \iff \lim_{x\to c^+} f(x) = L = \lim_{x\to c^-} f(x)$$