

Math 131A: Analysis

Discussion 8: Continuous and Uniformly Continuous Functions

1 Limits of Functions

Definition: Let $A \subset \mathbb{R}$, and $c \in A$ a limit point of A . Then, $\lim_{x \rightarrow c} f(x) = L$ if for every sequence $a_n \rightarrow c$ converging to c , we have $\lim_{n \rightarrow \infty} f(a_n) = L$.

Definition: The *right hand limit* of $f(x)$ at c is $\lim_{x \rightarrow c^+} f(x) = R$, and exists if for every sequence $a_n > c$ converging to c , $\lim_{n \rightarrow \infty} f(a_n) = R$.

The *left hand limit* of $f(x)$ at c is $\lim_{x \rightarrow c^-} f(x) = L$, and exists if for every sequence $a_n < c$ converging to c , $\lim_{n \rightarrow \infty} f(a_n) = L$.

1. For the following functions, what are the right and left limits at 0? What are the limits at $\pm\infty$?

(i) $f(x) = \frac{x^3}{|x|}$

(ii) $f(x) = \frac{\sin x}{x}$

(iii) $f(x) = \sin \frac{1}{x}$

2. Let $A \subset \mathbb{R}$. Let $f : A \rightarrow \mathbb{R}$ and $c \in \mathbb{R}$ a limit point of A . Prove the following:

$$\lim_{x \rightarrow c} f(x) = L \iff \lim_{x \rightarrow c^+} f(x) = L = \lim_{x \rightarrow c^-} f(x)$$

2 Continuous Functions

Definition: Let $A \subset \mathbb{R}$, $f : A \rightarrow \mathbb{R}$. f is *continuous* at a point $c \in A$ if for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$|x - c| < \delta \implies |f(x) - f(c)| < \epsilon.$$

f is continuous on A if it is continuous at every point $c \in A$.

Compositions of Continuous Functions: Let $f, g : A \rightarrow \mathbb{R}$ be continuous functions

- (i) $af + bg$ are continuous $\forall a, b \in \mathbb{R}$
- (ii) $f \cdot g$ is continuous
- (iii) $f \circ g$ is continuous where defined
- (iv) $\frac{f}{g}$ is continuous where defined

1. Prove that f is continuous at x_0 using ϵ - δ

- (a) $f(x) = x^2$, $x_0 = 2$
- (b) $f(x) = \sqrt{x}$, $x_0 = 0$
- (c) $x_0 = 0$,

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

- (d) $g(x) = x^3$, $x_0 \in \mathbb{R}$ arbitrary

2. Prove that f is discontinuous at x_0

- (a) $x_0 = 0$

$$f(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$$

- (b) $x_0 = 0$

$$g(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & x \neq 0 \\ g & x = 0 \end{cases}$$

3.

$$f(x) = \begin{cases} 1 & x = 0 \\ \frac{1}{q} & x = \frac{p}{q}, \gcd(p, q) = 1 \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Show that f is continuous at all irrational numbers and discontinuous at each rational number.

3 Uniform Continuity

Definition: Let $A \subset \mathbb{R}$, $f : A \rightarrow \mathbb{R}$ a function. f is *uniformly continuous* on A if for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in A$,

$$|x - y| < \delta \implies |f(x) - f(y)| < \epsilon.$$

1. Prove the following functions are uniformly continuous on the indicated set

- (a) $f(x) = 3x + 11$ on $\mathbb{R}$
- (b) $f(x) = x^2$ on $[0, 3]$
- (c) $f(x) = \frac{1}{x}$ on $[\frac{1}{2}, \infty)$

2. $g : (a, b) \rightarrow \mathbb{R}$ continuous. Prove g is uniformly continuous on (a, b) iff there exists a continuous function $f : [a, b] \rightarrow \mathbb{R}$ that extends g .