Math 61: Introduction to Discrete Structures
Discussion 3: Intro to Proofs 1/23/24, 1/25/24

1 Types of Proofs

I recommend checking out the Appendix! I tried to include some intuition about when/why one would use the different proof methods.

1.1 Direct or Contrapositive or Contradiction

1. Show that $\sqrt{2}$ is irrational (i.e. there does not exist a fraction p/q s.t. $(p/q)^2 = 2$). A similar proof should also be able to show that $\sqrt{3}$ is irrational.
2. Show that $\log_2(3)$ is irrational. (solution)
3. Prove that an rational number plus an irrational number must be irrational.
   (i) Do the same with "minus" in place of "plus"
   (ii) Is it true if we put "times" in place of "plus"? If not, describe the only exception, and prove it for non-exception cases
4. Fun puzzles one can do with the previous questions as background:
   (i) If a is irrational and b is rational, then can a^b be rational? What about irrational? (If yes, find explicit a, b that demonstrate that; if no, prove it.)
   (ii) If a and b are both rational, then can a^b be rational? What about irrational? (If yes, find explicit a, b that demonstrate that; if no, prove it.)
   (iii) If a and b are both irrational, then can a^b be rational? What about irrational? (If yes, find explicit a, b that demonstrate that; if no, prove it. HINT: the answer to both questions is YES, and the first 2 questions in this section are enough to find the explicit a, b.)
   (iv) If a is rational and b is irrational, then can a^b be rational? What about irrational? (If yes, find explicit a, b that demonstrate that; if no, prove it. HINT: the answer to both questions is YES, and the first 3 questions in this section are enough to find the explicit a, b.)
5. (This (essentially) is discussed in Appendix)
   (i) If n is an odd integer, prove that n^2 is an odd integer.
   (ii) If n^2 is an odd integer, prove that n is an odd integer.
6. (source) Color the plane $\mathbb{R}^2$ with 2 colors, say red and blue (so every point $(x, y) \in \mathbb{R}^2$ is colored either red or blue). **Claim:** there is a color C (equal to red or blue), so that for all distances $d > 0$, there exist 2 points of color C that are exactly distance d apart.
   (i) First, write out the negation of the **Claim**.
   (ii) Prove the **Claim** (by contradiction: show that the negation of the **Claim** leads to a contradiction). Hint: draw circles...
   (iii) Does your proof generalize to higher dimensions (e.g. coloring $\mathbb{R}^3$ by 3 colors, say red blue and green, **Claim:** there is a color C (equal to red or blue or green), so that for all distances $d > 0$, there exist 2 points of color C that are exactly distance d apart)? If not, find one that does.
7. (source) Suppose that every point in the plane is colored one of three colors. Then there are two points on the plane, exactly one unit apart, which have the same color.

1.2 Induction

1. (source) Prove that for all $n \in \mathbb{N}$, $5^{2n+1} + 2^{2n+1}$ is divisible by 7.

2. (source) Prove that, if $x > 0$ is any fixed real number, then $(1+x)^n > 1 + nx$ for all $n \geq 2$ (or for $n \geq 1$, we have $(1+x)^n \geq 1 + nx$). In particular, write out the claim you get by specializing to $x = 1$.

3. Prove that for all $n \in \mathbb{N}^+$,
$$\sum_{k=1}^{n} \frac{1}{k^2} < 2.$$
(Hint: it is EASIER to prove the STRONGER statement $\sum_{k=1}^{n} \frac{1}{k^2} < 2 - \frac{1}{n}$)

4. (source) Prove that for all $n \geq 2$,
$$\sqrt[n]{n} < \left(2 - \frac{1}{n}\right)$$

5. Prove that for any $n \in \mathbb{N}^+$, any $2^n \times 2^n$-chess board, with ANY square removed, can be tiled by 3-square L-shaped trominos (can rotate the pieces however you'd like)

6. For any interval $(a, b) \subseteq \mathbb{R}$, define its *length* to be $\ell((a, b)) := b - a$. Show that for any real numbers $x < y$ and any $n \in \mathbb{N}^+$, if the interval $[x, y]$ is *covered* by a collection of n open intervals $\{(a_i, b_i)\}_{i=1}^{n}$ (meaning $[x, y] \subseteq \bigcup_{i=1}^{n}(a_i, b_i)$), then
$$\ell([x, y]) \leq \sum_{i=1}^{n} \ell((a_i, b_i)) \quad \left(\text{i.e. } (y - x) \leq \sum_{i=1}^{n}(b_i - a_i)\right)$$

7. Suppose a grasshopper is sitting on the real line at $x = 0$, who can only hop to the right and land on integers. We choose $n-1$ distinct positive integers on which to place "mines" (that will kill the grasshopper if it lands on them). We then give the grasshopper n distinct positive integers $a_1 < a_2 < \ldots < a_n$ as stepsizes that the grasshopper can hop. Prove that it is possible for the grasshopper to pick the n possible stepsizes in some order, so that after making n hops of those stepsizes in that order, it gets to the final position $f := a_1 + \ldots a_n$ (on which we guarantee there is no mine), without dying. (solution)

1.3 Appendix ("philosophical" remarks about proof types + challenging content)

When to use the contrapositive to prove a statment? `https://math.stackexchange.com/a/650487/405572`

> Contraposition is often helpful when an implication has multiple hypotheses, or when the hypothesis specifies multiple objects (perhaps infinitely many).
>
> As a simple (and arguably artificial) example, compare, for x a real number:
>
> 1(a). If $x^4 - x^3 + x^2 \neq 1$, then $x \neq 1$. (Not easy to see without implicit contraposition?)
>
> 1(b). If $x = 1$, then $x^4 - x^3 + x^2 = 1$. (Immediately obvious.)

When to use the contrapositive to prove a statment? `https://math.stackexchange.com/a/638781/405572`:

> Here are some examples, hope they help. First an easy one.
>
> **Theorem**. Let n be an integer. If n is even then n^2 is even.
>
> **Proof** (outline). Let n be even. Then $n = 2k$ for some integer k, so $n^2 = 4k^2 = 2(2k^2)$, which is even.
>
> Now try the converse by the same method.
>
> **Theorem**. Let n be an integer. If n^2 is even then n is even.
>
> **Proof** (attempted). Suppose that n^2 is even, say $n^2 = 2k$. Then $n = \sqrt{2k}$ and so...???? This seems hopeless, $\sqrt{2k}$ does not look like an integer at all, never mind proving that it's even!
>
> Now try proving the converse by using its contrapositive.
>
> **Theorem**. Let n be an integer. If n^2 is even then n is even.
>
> **Proof**. We have to prove that if n is odd, then n^2 is odd. So, let $n = 2k+1$; then $n^2 = 4k^2+4k+1 = 2(2k^2+2k)+1$ which is odd. Done!
>
> I think the point here is that for the attempt at a direct proof we start with $n^2 = 2k$. This implicitly gives us some information about n, but it's rather indirect and hard to get hold of. Using the contrapositive begins with $n = 2k + 1$, which gives us very clear and usable information about n.
>
> Perhaps you could put a heuristic in the following form: "try both ways, just for a couple of steps, and see if either looks notably easier than the other".

What is the difference between a "proof by contradiction" and "proving the contrapositive"? `https://math.stackexchange.com/a/2106718/405572`:

> Take the assertion: *"When Mr So and So is happy, he sings."* The contrapositive asserts that *"Mr So and So does not sing so he's not happy"*. The negation asserts that *"There are days when Mr So and So is happy, yet he does not sing"*.
>
> I converted this example into logical notation *with quantifiers,* which makes the difference between negation and contrapositive more obvious.
>
> **Original statement:** Any day when Mr. So-and-so is happy is a day when he sings.
>
> $$\forall d \in D_{ays} : (H_{appy}(M_r) \to S_{ings}(M_r))$$
>
> **Contrapositive:** Any day when Mr. So-and-so does not sing is a day when he is not happy.
>
> $$\forall d \in D_{ays} : (\neg S_{ings}(M_r) \to \neg H_{appy}(M_r))$$
>
> **Negation:** There are some days (at least one day) when Mr. So-and-so is happy, but does not sing.
>
> $$\exists d \in D_{ays} : (H_{appy}(M_r) \wedge \neg S_{ings}(M_r))$$

What is the difference between a "proof by contradiction" and "proving the contrapositive"? `https://math.stackexchange.com/a/2263607/405572`:

> One difference is that proof by contrapositive only applies to propositions of the form $A \to B$ ("if-then propositions"). However not every proposition is a "if-then proposition", for example, consider the proposition, "$\exists x$ real s.t. for all p, q integers, $x \neq p/q$", there is no $\to$ inside that proposition, so it is not feasible to prove it by contrapositive.
>
> Proof by contrapositive:
>
> $(\neg B \to \neg A) \to (A \to B)$
>
> Proof by contradiction:
>
> $(\neg A \to (B \wedge \neg B)) \to A$

Why do mathematicians prefer direct proof or proof by contrapositive, over proof by contradiction? `https://mathoverflow.net/a/12400/112504`:

> Although the other answers correctly explain the basic logical equivalence of the two proof methods, I believe an important point has been missed:
>
> With good reason, we mathematicians prefer a direct proof of an implication over a proof by contradiction, when such a proof is available. (all else being equal)
>
> What is the reason? The reason is the fecundity of the proof, meaning our ability to use the proof to make further mathematical conclusions. When we prove an implication (p implies q) directly, we assume p, and then make some intermediary conclusions r1, r2, before finally deducing q. Thus, our proof not only establishes that p implies q, but also, that p implies r1 and r2 and so on. Our proof has provided us with additional knowledge about the context of p, about what else must hold in any mathematical world where p holds. So we come to a fuller understanding of what is going on in the p worlds.
>
> Similarly, when we prove the contrapositive (¬q implies ¬p) directly, we assume ¬q, make intermediary conclusions r1, r2, and then finally conclude ¬p. Thus, we have also established not only that ¬q implies ¬p, but also, that it implies r1 and r2 and so on. Thus, the proof tells us about what else must be true in worlds where q fails. Equivalently, since these additional implications can be stated as (¬r1 implies q), we learn about many different hypotheses that all imply q.
>
> These kind of conclusions can increase the value of the proof, since we learn not only that (p implies q), but also we learn an entire context about what it is like in a mathematial situation where p holds (or where q fails, or about diverse situations leading to q).
>
> With reductio, in contrast, a proof of (p implies q) by contradiction seems to carry little of this extra value. We assume p and ¬q, and argue r1, r2, and so on, before arriving at a contradiction. The statements r1 and r2 are all deduced under the contradictory hypothesis that p and ¬q, which ultimately does not hold in any mathematical situation. The proof has provided extra knowledge about a nonexistent, contradictory land. (Useless!) So these intermediary statements do not seem to provide us with any greater knowledge about the p worlds or the q worlds, beyond the brute statement that (p implies q) alone.
>
> I believe that this is the reason that sometimes, when a mathematician completes a proof by contradiction, things can still seem unsettled beyond the brute implication, with less context and knowledge about what is going on than would be the case with a direct proof.

Remarks: as the quote above indicates, the danger with proofs by contradiction is that you enter into this "illusory"/"ghostly" parallel universe that your ultimate goal is to show collapses in on itself, destroying/rendering useless all the things you proved inside. E.g. for more than 70 years, analytic number theorists have fought (without success) to collapse a particularly "real looking" parallel universe, the so-called "Siegel-zero universe". See TaoHeathBrown, TaoChowla, F&W "there is a ghost in the House of Prime Numbers" for further discussion.

Why do mathematicians prefer direct proof or proof by contrapositive, over proof by contradiction? `https://mathoverflow.net/a/386827/112504`:

> There is yet another reason for preferring proofs of the contrapositive to proofs by contradiction that has not been mentioned so far: It is harder to come up with wrong arguments that appear to be correct proofs when one tries to prove the contrapositive directly. When one attempts to prove something by contradiction, it is easy to declare victory if one can produce nonsense- even if the nonsense is the result of a mistake.

Video about induction (domino analogy) by me https://www.youtube.com/watch?v=vsqbyXjexUk&ab_channel=DanielRui. Strong induction has similar intuition: think of the domino analogy, but each successive dominon gets "heavier", and only gets knocked down by the combined weight of ALL the dominoes before it, not just the ONE immediately before it, like in standard induction.

Challenge Question (combining proofs by contradiction, strong induction, and casework! My Answer Here.)

(a) Show that any fraction $\frac{O}{E}$, where O is an odd integer and E is an even (non-zero) integer, can not be equal to an integer.

(b) Show that for all $n \in \mathbb{N}^+$,

$$\sum_{k=1}^{n} \frac{1}{k}$$

(when represented as a fraction in lowest terms/most simplified form) is of the form $\frac{[\text{odd}]}{[\text{even}]}$.

https://youtu.be/vQE6-PLcGwU?si=UmC4y6Ip4gb6Lvup&t=1166:

> So these partial sums creep up slower and slower smaller in smaller increments. On the way to infinity they pass every single one of the infinitely many integers and manage to miss them all by minuter and minuter amounts. Amazing isn't it?

Super Challenge: the proof below is not logically airtight, i.e. it makes a crucial claim that is unsubstantiated. Find the issue!

Claim: writing as a fraction in lowest terms, the nth partial sum $S_n := 1 + \frac{1}{2} + \ldots + \frac{1}{n}$ is of the form $\frac{[\text{odd}]}{[\text{even}]}$ $(n > 1)$.

Proof: by induction. **Base case:** $n = 2$, $S_2 = \frac{3}{2}$. By the inductive hypothesis, for $n > 2$, we now assume S_{n-1} is of the form $\frac{[\text{odd}]}{[\text{even}]}$. Let us also use the notation $S_{n-1} := \frac{O_{n-1}}{E_{n-1}}$.

- **Case 1:** n is odd, in which case

$$S_n = S_{n-1} + \frac{1}{n} = \frac{([\text{odd}] \cdot n) + (1 \cdot [\text{even}])}{[\text{even}] \cdot n} = \frac{\mathit{odd}}{\mathit{even}},$$

  (using that odd times odd is odd, plus even is still odd) which remains true even after reducing the fraction to lowest terms.

- **Case 2:** n is even, say $n = 2^k \cdot [\text{odd}]$. It will also be important to write the even denominator E_{n-1} of S_{n-1} as $2^m \cdot [\text{odd}]$. Then,

$$\begin{aligned} S_n = S_{n-1} + \frac{1}{n} = \frac{[\text{odd}]}{E_{n-1}} + \frac{1}{n} &= \frac{([\text{odd}] \cdot n) + (1 \cdot E_{n-1})}{E_{n-1} \cdot n} \\ &= \frac{([\text{odd}] \cdot 2^k \cdot [\text{odd}]) + (2^m \cdot [\text{odd}])}{2^k \cdot [\text{odd}] \cdot 2^m \cdot [\text{odd}]} \\ &= \frac{(2^k \cdot [\text{odd}]) + (2^m \cdot [\text{odd}])}{2^k \cdot 2^m \cdot [\text{odd}]}. \end{aligned}$$

  WLOG, $2^k \leq 2^m$ (otherwise, just swap the variable names in the previous expression), and clearing 2^k from top and bottom, get

$$S_n = \frac{(1 \cdot [\text{odd}]) + (2^{m-k} \cdot [\text{odd}])}{1 \cdot 2^m \cdot [\text{odd}]}.$$

  We see that we're done, as long as $2^k \neq 2^m$. Indeed, this must be true:

  - **Case 2.1:**: if $n = 2^k \cdot 1$, then S_{n-1} is the sum of fractions with at most a power of 2^{k-1} in the denominator, so it must itself have at most a power of 2^{k-1} in the denominator. In the notation above, we have $2^m \leq 2^{k-1}$.
  - **Case 2.2:**: if $n = 2^k \cdot [\text{odd } \# \geq 3]$, then S_{n-1} has already seen $\frac{1}{2^k \cdot 2}$, so the denominator E_{n-1} of S_{n-1} has at least a power of 2^{k+1}. In the notation above, we have $2^m \geq 2^{k+1}$.