Math 61: Introduction to Discrete Structures
Discussion 4: Induction and Equivalence Relations

1 Proof by Induction

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1. Use induction to show for all whole numbers $n \geq 1$,

$$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$$

2. Let $r \in \mathbb{R}$ such that $0 < |r| < 1$. Use induction to show $\forall n \geq 1$,

$$r^0 + r^1 + \cdots + r^n < \frac{1}{1-r}$$

3. A *tromino* is a tile consisting of three 1x1 squares aligned edge to edge. Use induction to show that any $2i \times 3j$ grid can be tiled with trominoes, $\forall i \geq 1, j \geq 1$.

4. Draw n lines in the plane $\mathbb{R}^2$ such that no two lines are parallel, and no three lines intersect at a point. Use induction to show that these lines divide the plane into

$$R(n) = \frac{n^2 + n + 2}{2}$$

regions.

Hint: How many regions are added by drawing another line?

2 Equivalence Relations

1. Which of the following are equivalence relations on $\{1, 2, 3, 4, 5\}$?

   (i) $\{(1,1),(2,2),(3,3),(4,4),(5,5),(1,3),(3,1),(3,4),(4,3)\}$

   (ii) $\{(1,1),(2,2),(3,3),(4,4),(5,5),(1,3),(3,1)\}$

   (iii) $\{(1,1),(2,2),(3,3),(4,4),(1,4),(4,1),(3,4),(4,3),(1,3),(3,1)\}$

   (iv) $\{(1,d),(2,d),(4,a)\}$

   (v) $\{(x,y) : 4 \text{ divides } x-y\}$

2. Let X be the set of all UCLA students, and define the relation $x\ R\ y$ if the students x and y have a major in common. (Recall that UCLA students may choose more than one major, and assume that all UCLA students have declared at least one major.) Is R an equivalence relation?

3. Let Y be the set of all USC students, and define the relation $x\ S\ y$ if the students x and y have a major in common. (Suppose that USC, being much worse than UCLA, only allows students to choose one major, and assume that all USC students have declared at least one major.) Is S an equivalence relation?

4. How many equivalence relations are there on $\{1, 2, 3\}$?

5. Let $X = \{1, 2, \ldots, 10\}$. Define a relation R on $X \times X$ by $(a, b)\ R\ (c, d)$ if $ad = bc$.

   (a) Show that R is an equivalence relation on $X \times X$.

   (b) List one member of each equivalence class.

   (c) Describe this equivalence class in words.

6. Let R_1, R_2 be equivalence relations on X.

   (a) Show that $R_1 \cap R_2$ is an equivalence relation on X.

   (b) Describe the equivalence classes of $R_1 \cap R_2$ in terms of the equivalence classes of R_1, R_2.