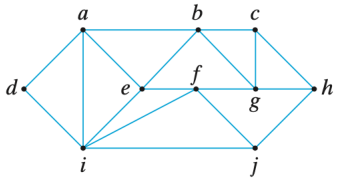
Math 61: Introduction to Discrete Structures
Discussion 8: Graph Theory (Hamiltonian Paths & Adjacency Matrices)

1 Graph Theory

1. Let G be a connected graph. Suppose that an edge e is part of some cycle. Show that $G - \{e\}$ is still connected.
2. A closed path is a path from v to v. Show that a connected graph G is bipartite if and only if every closed path in G has even length.

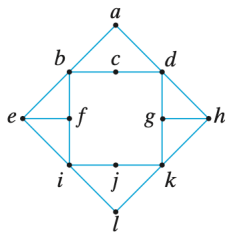
2 Hamiltonian Cycles

1.



1. Show that the graph above has a Hamiltonian Cycle

3.



2. Show that the graph above does not contain a Hamiltonian Cycle.
3. Give an example of a graph that has an Eulerian cycle but contains no Hamiltonian cycle.
4. Let G be a bipartite graph with disjoint vertex sets V_1 and V_2. Show that if G has a Hamiltonian cycle, V_1 and V_2 have the same number of elements.
5. Show that if $n \geq 3$, the complete graph on n vertices K_n contains a Hamiltonian cycle.

6. (Challenge Problem) The Hypercube Q_n has vertex set $\{0,1\}^n$ and two vertices share an edge if they differ in exactly one coordinate. Prove that Q_n contains a Hamiltonian cycle for $n \geq 2$. (Hint: Prove by induction. For inductive step break up Q_n into two disjoint cubes Q^0_{n-1}, Q^1_{n-1} where last coordinate is fixed to be 0 or 1, respectively. WLOG have the cycles in each take the 'same' path).

3 Adjacency Matrix

---

1. Suppose that a graph has an adjacency matrix of the form below where all entries in A' and A'' are 0.

$$A = \left(\begin{array}{c|c} & A' \\ \hline A'' & \end{array} \right),$$

   What must the graph look like?

2. Let A be an adjacency matrix of a graph.Why is A^n symmetric about the main diagonal for every positive integer n?