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Classification of Zamolodchikov Periodic Cluster Algebras

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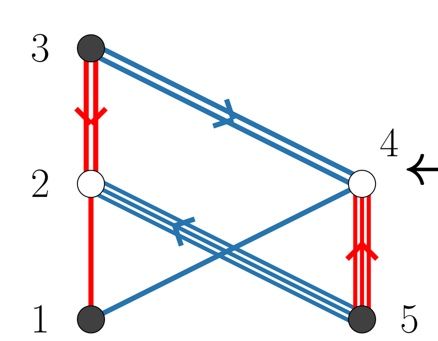
Bipartite Dynamics

Definitions:

- A bipartite matrix B is *recurrent* if $\mu_\circ(B) = \mu_\bullet(B) = -B$, where $\mu_\circ$ and $\mu_\bullet$ is cluster mutation at all white and all black indices, respectively. Explicitly, B is recurrent if we can write $B = \Gamma + \Delta$, where Γ, Δ are commuting skew-symmetrizable matrices with the orientations $\Gamma_{\circ,\bullet} > 0$ and $\Delta_{\bullet,\circ} > 0$.
- A bipartite recurrent $n \times n$ matrix B has an associated T-*system*, an infinite family of elements $T_i(t), i \in \{1, \ldots, n\}$ which satisfies the following exchange relations.
$$T_k(t-1)T_k(t+1) = \prod_j T_j(t)^{|\Gamma_{jk}|} + \prod_j T_j(t)^{|\Delta_{jk}|}$$
- A bipartite recurrent matrix B is *Zamolodchikov periodic* if for some $N \in \mathbb{N}$, $T_i(t+2N) = T_i(t)$ for every $i \in \{1, \ldots, n\}$.

Although Zamolodchikov periodicity was first observed in a physics setting, Fomin–Zelevinsky proved periodicity for Dynkin diagrams by identifying the elements of the T-system with cluster variables of a cluster algebra following a certain mutation sequence $\mu_\circ\mu_\bullet\mu_\circ\ldots$ called the *bipartite belt*. Keller later proved it for tensor products of two Dynkin diagrams using categorification.



$\longleftrightarrow$

	1	2	3	4	5
1	0	−1	0	1	0
2	1	0	2	0	−3
3	0	−1	0	1	0
4	−1	0	−2	0	3
5	0	1	0	−1	0

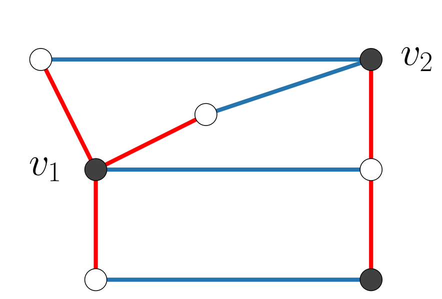
Figure 1. A depiction of a bipartite recurrent matrix, with Γ colored red and Δ colored blue.

Dynkin Biagrams

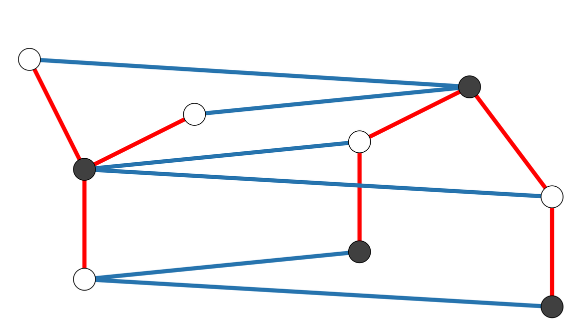
Definitions:

- A *Dynkin biagram* is a tuple (Γ, Δ) of two (not necessarily irreducible) $n \times n$ Coxeter adjacency matrices such that their sum $B = \Gamma + \Delta$ is bipartite and Γ, Δ share no nonzero entries.
- A Dynkin biagram (Γ, Δ) is *admissible* if the associated Cartan matrices $C_\Gamma = 2I - \Gamma$, $C_\Delta = 2I - \Delta$ commute, or equivalently if Γ and Δ commute.
- A Dynkin biagram whose associated Cartan matrices are simply-laced is called an *ADE bigraph*.

Admissible ADE bigraphs were classified by Stembridge in his study of admissible W-cells, and were found to be Zamolodchikov periodic by Galashin–Pylyavskyy.



Not admissible



Admissible

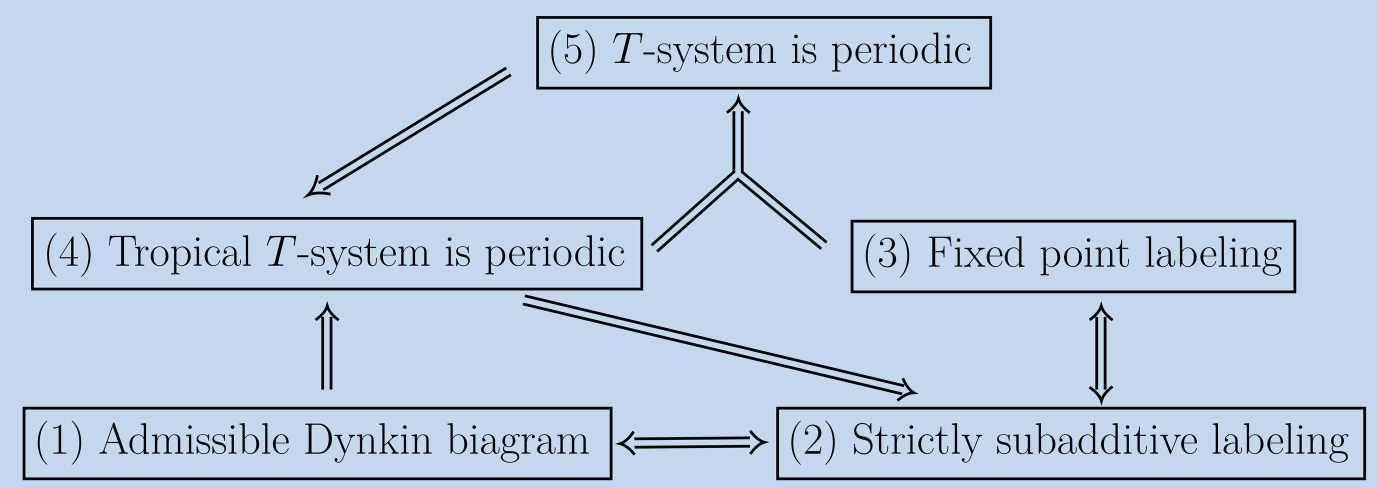
Main Result

Theorem: The Zamolodchikov periodic cluster algebras are in natural bijection with pairs of commuting (not necessarily reduced or simply-laced) Cartan matrices.

The proof of this theorem is broken up into two parts:

Theorem: Pairs of commuting Cartan matrices (equivalently, admissible Dynkin biagrams) are classified into 35 infinite families and 25 exceptional types.

Theorem: The Zamolodchikov periodic cluster algebras are exactly the admissible Dynkin biagrams.



Operations on Dynkin Biagrams

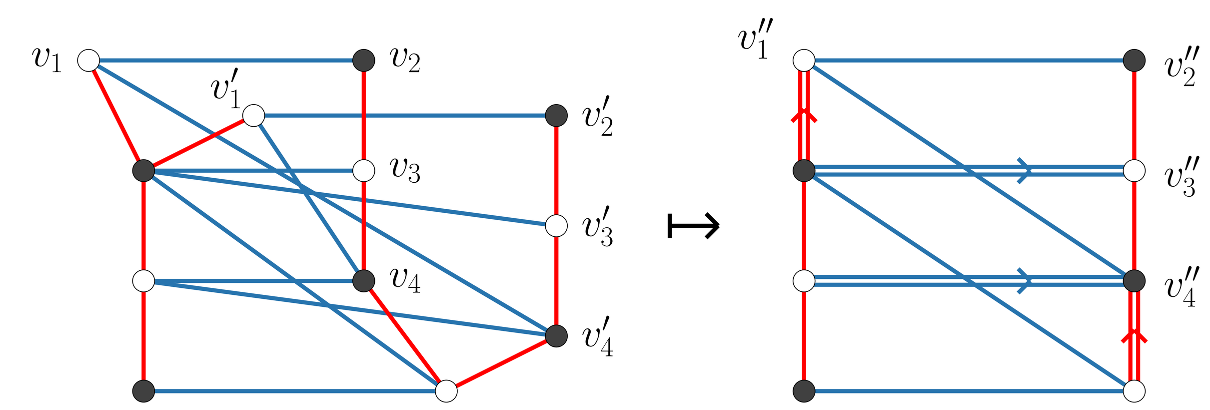


Figure 2. The folding of an ADE bigraph into a Dynkin biagram.

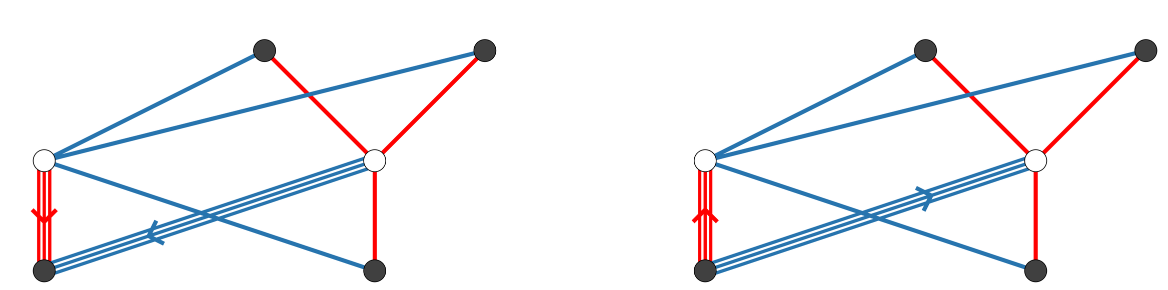


Figure 3. Two Dynkin biagrams corresponding to a matrix and its transpose

Proposition: Every Dynkin biagram can be obtained from an ADE bigraph via folding and taking transpose.

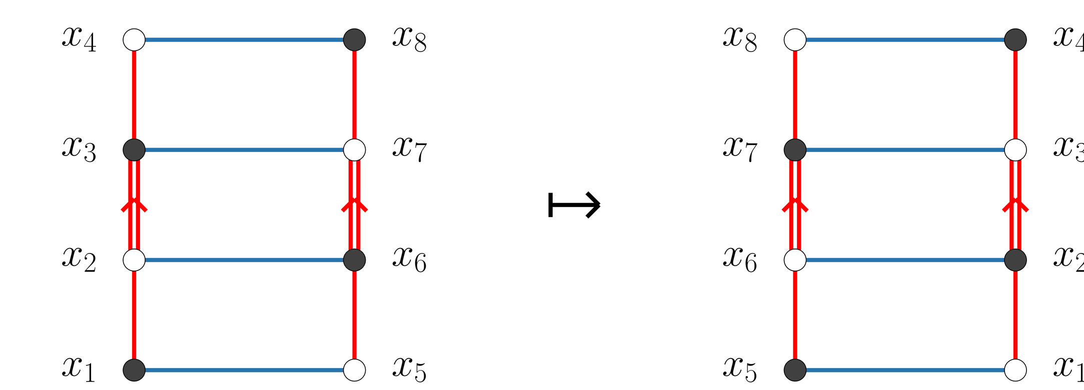
Key Lemma: Zamolodchikov periodicity is preserved under folding and taking transpose.

It is a standard fact that folding commutes with cluster mutations and thus preserves Zamolodchikov periodicity. On the other hand, taking transpose does not commute with cluster mutations, and thus it is surprising that it preserves Zamolodchikov periodicity.

More on Periodicity

The period of each admissible Dynkin biagram (Γ, Δ) divides $2(h_\Gamma + h_\Delta)$, where h_Γ, h_Δ are the Coxeter numbers associated with Γ, Δ, respectively. As a result of the classification, we were able to determine the behavior of the cluster variables at the half-period.

Theorem: Let $T(t)$ be the T-system associated to (Γ, Δ). Then
$$T_i(t + h_\Gamma + h_\Delta) = T_{\sigma(i)}(t) \quad \forall i \in \{1, \ldots, n\}$$
for some automorphism σ of (Γ, Δ) of order at most two. This permutation is bipartite color preserving when $h_\Gamma + h_\Delta$ is even, and color reversing otherwise.



Colored Mutations

The tropical T-system with initial labeling λ has the exchange relation
$$\boldsymbol{t}_k^\lambda(t-1) + \boldsymbol{t}_k^\lambda(t+1) = \max\left(\sum_j |\Gamma_{ij}|\boldsymbol{t}_j^\lambda(t), \sum_j |\Delta_{ij}|\boldsymbol{t}_j^\lambda(t)\right),$$
which can be classified as a Γ-mutation or a Δ-mutation, depending on the evaluation of the max function.

Conjecture 1: Let (Γ, Δ) be an admissible Dynkin biagram with r vertices. For any initial labeling λ, the number of Γ-mutations in one period is $h_\Gamma \cdot r$, and the number of Δ-mutations in one period is $h_\Delta \cdot r$.

Conjecture 2: Let (Γ, Δ) be an admissible biagram with r vertices where the components of Γ are affine Dynkin diagrams and the components of Δ are Dynkin diagrams. The number of Δ-mutations is bounded above by $\frac{1}{2}(h_\Delta \cdot r)$.

The space of initial labelings $\lambda \in \mathbb{R}^r$ can be partitioned into chambers based on the sequence of colored mutations produced by λ. These represent domains of linearity of the tropical T-system.

Conjecture 3: Let Λ be a Dynkin diagram whose B-matrix is full rank r. Consider the Dynkin biagram $\Lambda \otimes A_1$. The space of initial labelings $\mathbb{R}^r$ is partitioned into c_Λ distinct chambers, where c_Λ is the number of seeds in the cluster algebra of type Λ.

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